Stability Analysis of Orbital Motions around Uniformly Rotating Irregular Asteroids

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Through the approach of periodic orbits, stability of the orbital motions around uniformly rotating irregular asteroids is analyzed. For the planar case, studies show that the phase space structure is quite different from that of the two-body problem when the 2OD terms are large, and stable orbital motions interior of the 1:1 resonance generally do not exist. For the three-dimensional orbits, studies show that the motions with orbit inclinations close to the critical values are unstable, thus are able to escape from or collide with the asteroid following a non-planar route. Taking Eros as an example and considering higher order non-spherical terms, some extraordinary orbits are found, such as orbits with orbital plane co-rotating with the asteroid, orbits are "quasi-stationary" in space, and frozen orbits with argument of perigee different from 0, 90, 180, and 270 degrees.

Key Words: asteroid, periodic orbit, resonance, stability, critical orbit inclination

1. Introduction

Asteroids are remnants of the early solar system and are targets of many observation/search programs from both the ground and the space. With the success of several past space missions, there is a growing interest on sending new probes to these primitive objects, such as the two on-going missions Hayabusa-2[1)] and OSIRIS-Rex[2)], and also some proposed or planned missions.

Different major bodies in the solar system, asteroids are usually much smaller and irregularly-shaped. This makes their gravity much weaker and irregular compared with those of the major bodies, which cause difficulties to studying the probe's orbital motion, for two problems: (1) the traditional analytical satellite theory based on perturbation theory converges slowly or fail due to the relatively large non-spherical perturbations. (2) the relatively small gravity makes other perturbative forces such as the solar radiation pressure much larger. This paper focuses on the first problem.

Through the approach of periodic orbits in the asteroid's body-fixed frame (which can also be viewed as resonance orbits in the inertial frame), we qualitatively describe how the phase space structure and the stability of the orbital motions around asteroids are affected by non-spherical terms. For the planar orbital motions and highly irregular asteroids, we find that the phase space structure is quite different from that of the two-body problem, and generally no stable motions interior of the 1:1 resonance can be found. For the three-dimensional motion, we find that orbits with orbit inclinations close to the critical values are unstable.

To make the results of the current paper more general, most of the work is carried out in the 2OD gravity field. However, results in the current work can be used as a step stone to the results in the gravity field of specific asteroids. At the end of the work, taking the asteroid Eros and an example, and also considering a gravity with higher order non-spherical terms, some interesting periodic orbits are reported. This conference paper is a brief summarize of the work presented in Ref 3) and 4). Readers who are interested in the results can find more details in these two references.

2. Equations of Motion

In the body-fixed frame of a uniformly rotating asteroid, the orbit of a small body follows

$$\ddot{\mathbf{r}} + 2n_a\left(-\dot{y}, \dot{x}, 0\right)^T - n_a^2\left(x, y, 0\right)^T = \frac{\partial V}{\partial \mathbf{r}} \tag{1}$$

where n_a is the rotation speed of the asteroid, and V is the minus of the asteroid's potential[5)]. Truncated at the second order, V takes the form of

$$V = \frac{1}{r} - \frac{J_2}{r^3}\left(\frac{3}{2}\frac{z^2}{r^2} - \frac{1}{2}\right) + \frac{J_{22}}{r^5}\left(x^2 - y^2\right) \tag{2}$$

To attribute physical meaning to the 2OD terms J_2 and J_{22}, we use an ellipsoid shape model with three semi-axes as $a \geq b \geq c$ and a constant density of 2.5g/cm^3 for the asteroid. We have

$$C_{20} = \frac{1}{5R_e^2}\left(c^2 - \frac{a^2+b^2}{2}\right), \qquad C_{22} = \frac{1}{20R_e^2}\left(a^2 - b^2\right) \tag{3}$$

Instead of using a value representing the size of the asteroid for the reference radius *Re*, we use the height of the synchronous orbit

$$R_e = r_{syn} = \left(GM/\omega_a^2\right)^{1/3} \tag{4}$$

which is obtained by assuming the asteroid as a particle. Eq. (1) admits an integral of the form

$$2\Omega - v^2 = C \tag{5}$$

where C is called as the Jacobi constant. Eq. (1) also admits four equilibrium points outside of the asteroid, two of which lie at the long axis and the other two lie at the short axis. Viewing the equilibrium points in the inertial frame, they are actually orbits that are in 1:1 resonance with the asteroid's rotation.

One remark is, only the 2OD gravity of the asteroid is considered, although the ellipsoid shape model for the asteroid is taken.

3. Stability of the Planar Motion

3.1. Poincaré section

For the planar case, Poincaré maps serve as a useful tool to separate stable orbits from unstable ones. Fig. 1 shows the Poincaré maps for the Jacobi constant of the equilibrium point at the asteroid's long axis. The asteroid has a shape parameter as *a:b:c* = 1km: 0.65km: 0.4km. The rotation period is 36h, 30h, 24h and 20h.

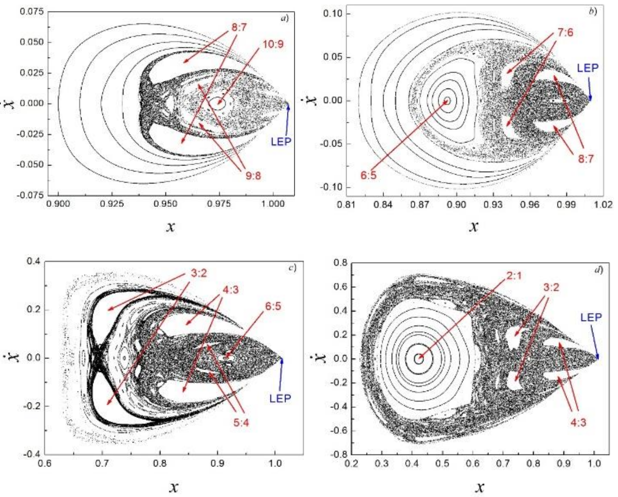

Fig. 1. Poincaré maps for a same asteroid but with different rotation periods.

One obvious feature of Fig. 1 is that there are many resonance islands in Fig. 1, and these islands gradually disappear when the asteroid rotates faster. The centers of the resonance islands are actually periodic orbits in the asteroid's body-fixed frame. The problem of the Poincaré maps such as the one in Fig. 1 is that we have to fix one Jacobi constant to generate it. If the Jacobi constant changes, we may have a completely different map. The reason is that we peer the dynamics of the system from different slices of the energy but not globally. On the contrary, we are able to globally describe how the resonance orbits are organized in the phase space, via the approach of periodic orbits in the asteroid's body-fixed frame, even when the non-spherical perturbations are very large.

3.2. Families of periodic orbits

In this study, there are three kinds of periodic orbits. Two have member with near-circular shapes in both the asteroid's body-fixed frame and the inertial frame. They are denoted as Family I and II in this study. The other one has members of eccentric orbits in the inertial frame, and is denoted as Family III. Family I is interior of the 1:1 resonance and Family II is outside of the 1:1 resonance. In the unperturbed two-body problem, the relation between these periodic families (we use the terminology "genealogy") is shown in Fig. 2. The ordinate is the Jacobi constant and the abscissa is the period.

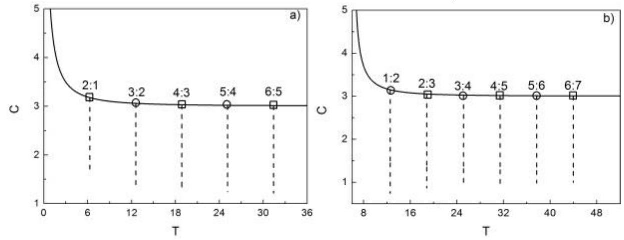

Fig. 2. Genealogy between the three periodic families on the *T*-*C* plane for the unperturbed two-body problem. Family I are II are denoted as solid lines and Family III is denoted as dashed lines.

In the presence of the 2OD terms, the genealogy is distorted by perturbations, as shown by Fig. 3 and Fig. 4.

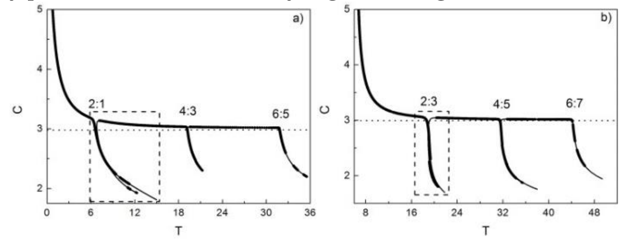

Fig. 3. Genealogy between periodic families for an asteroid with *a:b:c* = 1km: 0.9km: 0.8km and a rotation period of 12h.

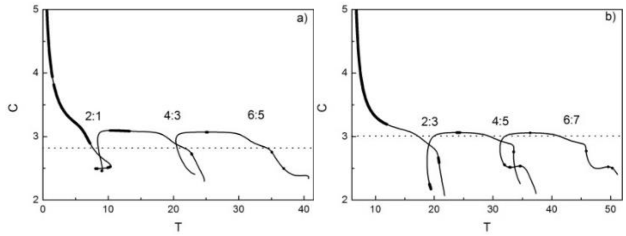

Fig. 4. Genealogy between periodic families for an asteroid with *a:b:c* = 1km: 0.6km: 0.4km and a rotation period of 12h.

Comparing Fig. 2 with Fig. 3-4 shows that each resonance splits into two branches, but the genealogy between periodic families generally remain unchanged although distorted from that of the 2BP. This means it's still proper to view the orbital motions close to the asteroid as perturbed Keplerian orbits. However, when the asteroid is even more elongated, the genealogy of periodic families is completely different from that of the 2BP. Fig. 5 shows only two periodic families for the case of *a:b:c* = 1km: 0.3km: 0.2km.

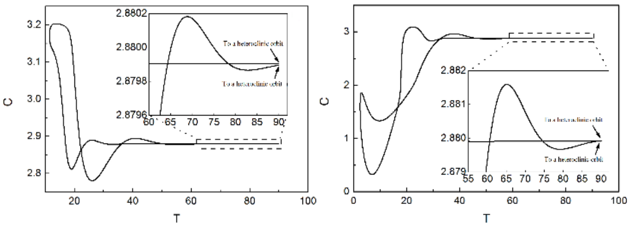

Fig. 5. Genealogy between periodic families for an asteroid with *a:b:c* = 1km: 0.6km: 0.4km and a rotation period of 12h.

For clarity, many other periodic families that are also in the same region of the *T-C* plane are not shown in Fig. 5. Difference between Fig. 5 and Fig. 2-4 indicates that maybe it is no longer valid to view the orbital motions as perturbed Keplerian orbits anymore.

3.3. Stability analysis

Thick lines in Fig. 3 and Fig. 4 indicate the stable orbits in the families. With the increase of the 2OD terms, stable region quickly shrinks on the *T-C* plane. For example, in Fig. 4, except some stable 2:1 resonance orbits, generally no stable orbits exist in each resonance. This indicates that the stable resonance islands in Fig. 1 no longer exists, no matter what the Jacobi constant is. A qualitative picture is given in Fig. 6. The gray ring in Fig. 6 indicates a chaotic region enveloping the 1:1 resonance. In this chaotic region, generally no stable resonance orbits exist except the 1:1 one. When the asteroid becomes more elongated (a) or rotates fasters, the width of this region increases and may eventually touches the surface of the asteroid. In such cases, no stable orbits interior of the 1:1 resonance can be found.

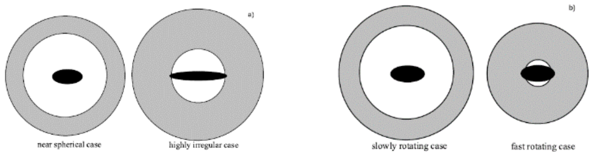

Fig. 6. Two intuitive pictures showing the process of gradual disappearance of stable regions interior of the 1:1 resonance.

One remark is: stability of retrograde periodic orbits shows much better robustness than prograde periodic orbits with respect to non-spherical perturbations.

4. Stability of the Three-Dimensional Motion

4.1. The secular resonance

Studies show that the problem of critical orbit inclination plays an important role in determining the stability of three -dimensional orbits. When orbits have inclinations close to the critical values $i_c^{pro}=\cos^{-1}1/\sqrt{5}$ or $i_c^{retro}=\pi-i_c^{pro}$, there is no secular change to the argument of periapsis ω, i.e., $\dot{\omega}=0$. We view the problem in this work as the secular resonance which happens between the longitude of the ascending node Ω and the angle $\tilde{\omega}=\omega+\Omega$, i.e., $\dot{\omega}=\dot{\tilde{\omega}}-\dot{\Omega}=0$. Since periodic orbits are resonance orbits in the inertial frame, when the three-dimensional periodic orbits have inclinations close to the critical values, the orbital resonance can be taken as overlapping with the secular resonance, and this causes chaos and instability of the three-dimensional motion. Similar phenomena have been observed in many problems of celestial mechanics[6-8], including the perturbed motions by the non- spherical gravity terms[9-10].

The mechanism is qualitatively described as follows. Take the 2:1 resonance as an example. Actually there are a series of resonance angle for this resonance, as given by Eq. (6). The difference between these resonance angles is the coefficients of the argument of periapsis. Denote these resonance angles as sub-resonances. The distance between these sub-resonances is denoted as Δa.

$$\begin{cases} \vdots \\ M+2(\Omega-\theta_a)-2\omega \\ M+2(\Omega-\theta_a)-\omega \\ M+2(\Omega-\theta_a) \\ M+2(\Omega-\theta_a)+\omega \\ M+2(\Omega-\theta_a)+2\omega \\ \vdots \end{cases} \tag{6}$$

Considering the fact that

$$\dot{\omega}\sim-\frac{3J_2}{4a^{7/2}\left(1-e^2\right)^2}\left(1-5\cos i\right)$$

we have $\Delta a\to 0$ when $\dot{\omega}\to 0$, which happens when the orbit inclinations are close to the critical values. This causes the overlap of different sub-resonances and the chaos of this resonance, a process as illustrated by Fig. 7. In the left frame, this orbital resonance does not overlap with nearby resonances. As a result, even the orbital motion is chaotic within this resonance, the chaos is not global, i.e., the orbit does not escape. However, if this resonance also overlaps with nearby resonances, then the motion becomes globally chaotic and may eventually escape this resonance, as shown by the right frame of Fig. 7.

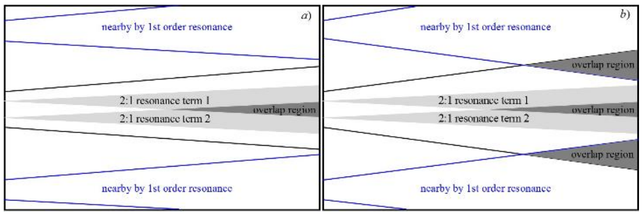

Fig. 7. Two intuitive pictures showing overlap between sub-resonances (left) and different resonances (right).

4.2. Families of periodic orbits

In the unperturbed two-body problem, three-dimensional periodic families connect vertical bifurcation orbits of Family III with those of Family I or Family II. An intuitive picture is shown in Fig. 8.

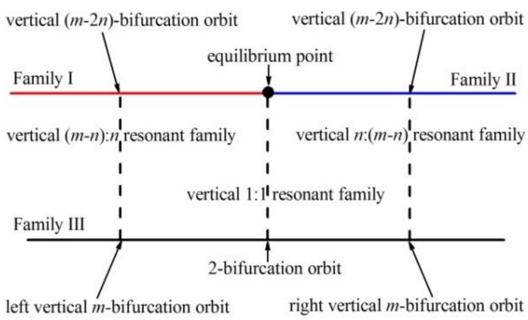

Fig. 8. An illustrative picture showing the genealogy between the three-dimensional periodic families (denoted as "vertical (m-2n:n) resonant family" or "n:(m-n) resonant family" in the figure), and Family I, II, and III.

However, in the presence of the 2OD terms, this genealogy may be broken apart by families of eccentric frozen orbits. In the perturbed case, each three-dimensional periodic family splits into two branches. We only take the 2:1 resonance as example. Fig. 9 shows the genealogy of this family for three asteroids with shape parameters *a:b:c* as 1km: 0.99km: 0.4km (a, b), 1km: 0.95km: 0.4km (c, d), and 1km: 0.6km: 0.4km (e, f). We can see that the *T-C* curves of these three-dimensional periodic families are distorted from vertical lines on the *T-C* plane by the 2OD terms. Moreover, each branch of the three-dimensional periodic family is broken apart by families of eccentric frozen orbits, such as the ones enveloped in the dashed squares in Fig. 9a.

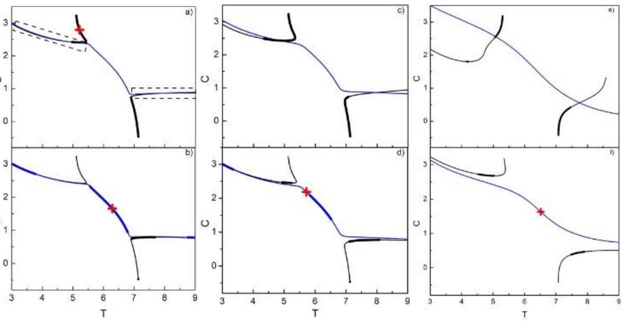

Fig. 9. Genealogy of three-dimensional 2:1 resonance family in the presence of 2OD terms. The upper and lower three frames are for different branches of the 2:1 resonance. From left to right, the frames correspond to asteroids with different shapes.

One remark is that the genealogy of the three-dimensional periodic families is not always broken apart by families of eccentric frozen orbits. Shown in Fig. 9 is just a particular example for the 2:1 resonance. For other resonances, there are cases where one branch is broken apart and the other branch remains a single family, or cases where both branches remain a single family. Moreover, the genealogy of the periodic families may change for different combinations of the J_2 and J_{22} terms.

4.3. Stability analysis

In section 4.1, stability property of spatial orbital motions is already introduced. In this section, we take specific examples to support the arguments in section 4.1.

(1) In the case of Fig. 9a and b where the J_{22} term of the asteroid is very small, the stable segment on the *T-C* curve with orbit inclinations between i_c^{pro} and i_c^{retro} exists. Due to the extremely small J_{22} term, the width of the sub-resonances of the 2:1 resonance which is directly proportional to J_{22} is small. They do not overlap with each other, and allows stable regions far away from i_c^{pro} and i_c^{retro} exist. Fig. 10a shows the time history curve of the resonance angles for two stable orbits denoted as crosses in Fig. 9a and b.

(2) In the case of Fig. 9c and d where the J_{22} term of the asteroid becomes larger, starting from both ends, the stable region between i_c^{pro} and i_c^{retro} shrinks. Fig. 10b shows the time history curve of one orbit which was stable in Fig. 9b but now becomes unstable due to the influence of the secular resonance. This orbit is chaotic and exhibits large vibrations in its orbit eccentricity, but this orbit does not escape. This corresponds to the case shown by the left frame of Fig. 7.

(3) In the case of Fig. 9e and f where the J_{22} term of the asteroid is much larger, the stable region between i_c^{pro} and i_c^{retro} completely disappears. In this case, not only the sub-resonances of the 2:1 one overlap with each other, but also the 2:1 resonance overlaps with nearby ones and cause the global instability of the orbits. One example is shown in Fig. 10c, with its positions on the *T-C* curve denoted as a cross in Fig. 9f. In Fig. 10c, two curves are given. One is for the orbit eccentricity, which shows that the eventual fate of the orbit is to escape. The other is for the orbit inclination, which indicates that the escape route follows an orbit inclination close to i_c^{pro} .

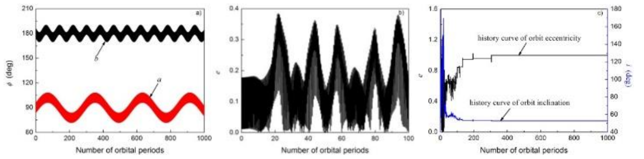

Fig. 10. Left: time history curve of two stable orbits denoted as crosses in Fig. 9a and b; Middle: time history curve of the orbit eccentricity for an unstable orbit denoted as a cross in Fig. 9d; Right: time history of the orbit eccentricity and the orbit inclination for an unstable orbit denoted as a cross in Fig. 9f.

5. A Case Study

Above work is carried out in the 2OD gravity. The results can help us qualitatively describe the global dynamics around asteroids, but generally can be directly applied to specific asteroids. However, as already being stated previously, results of the 2OD gravity can serve as good initial seeds to be continued to the results for specific asteroids. In this section, we take the asteroid Eros as an example to show some results. Details of the periodic families are omitted here. Readers can found them in Refs 3) and 4). We only report some interesting periodic orbits found here. The results are presented in the 4th order and 4th degree gravity of Eros.

5.1. Out-of-plane stable orbits co-rotating with asteroid

We find that there are stable out-of-plane periodic orbits with their orbital plane co-rotating with Eros, such as the two ones shown in Fig. 11a. Both orbits are retrograde. For fast rotating irregular asteroids such as Eros, these orbits are of no practical uses because they already collide with the asteroid's surface. However, for slowly rotating asteroids, these stable out-of-plane orbits are of practical uses because they are outside the asteroid. For example, if we artificially set Eros's rotation period as 48h, Fig. 11b shows two such example orbits. Existence of these periodic orbits is due to the secular resonance $\dot{\Omega} = n_a$ which happens between the precession rate of the orbital ascending node and the asteroid's rotation. These orbits are interesting because they can be used to repeatedly visit certain regions of the asteroid.

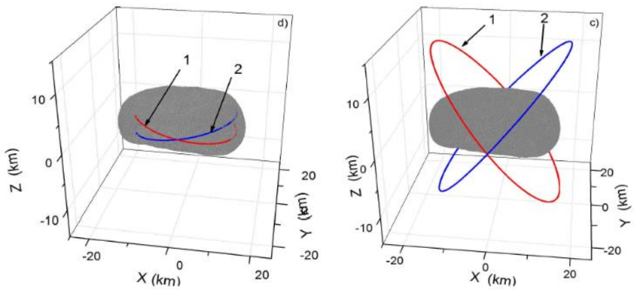

Fig. 11. Left: two stable out-of-plane orbits around Eros when its rotation period takes the current true value of 5.27h. Right: two stable out-of-plane orbits when Eros's rotation period is set as 48h.

5.2. Quasi-stationary orbits

We also find periodic orbits that are not only in the resonance $\dot{\Omega} = n_a$ as the one in Fig. 11, but also in the resonance $\dot{\Omega} + \dot{\omega} - \dot{\lambda} = -\dot{M} = 0$. This means that the mean orbital frequency of the orbit is zero, which also means that the orbit does not circle the asteroid in one period. So viewing in the inertial frame, the orbit is "quasi-stationary". Fig. 12 shows such an example in the 4th order and 4th degree gravity of Eros. We notice that part of the orbits lies within the Brillouin sphere where the 4th order and 4th degree gravity is no longer valid. This makes the results in Fig. 12 doubtful. Existence of such orbits in the asteroid's real gravity still requires further work.

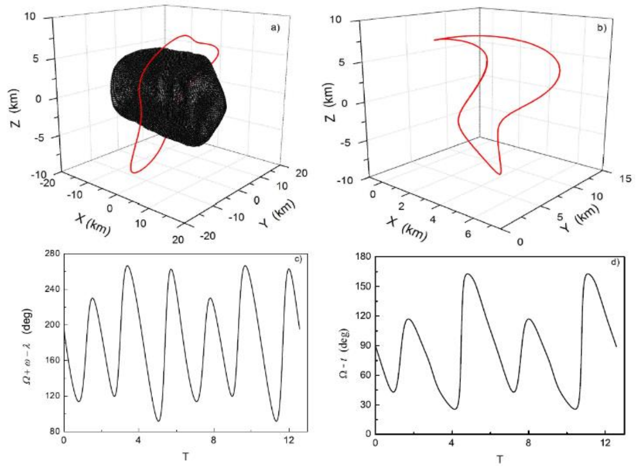

Fig. 12 A "quasi-stationary" orbit around the asteroid Eros

6. Conclusion

This paper describes the global dynamics around uniformly rotating irregular asteroids, in a 2OD gravity, through the approach of periodic orbits, in the body-fixed frame of the asteroid. For the planar case, studies show that for extremely elongated asteroids, the phase space structure is different from that of the 2BP, indicating that it is no longer to treat the orbital motion close to the asteroid as Keplerian orbits. For highly elongated asteroids, orbital motions interior of the 1:1 resonance are generally unstable. For three-dimensional orbits, their stability is influenced by the secular resonance. Near circular orbits with orbit inclinations close to critical values are generally unstable.

Acknowledgments

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